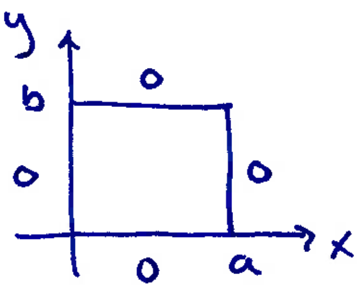
2D Case

$u_t = k(u_{xx} + u_{yy})$ $\quad 0<x<a \quad 0<y<b$



$u(x,0)=0$ $\quad$ ~~$u(x,0)=$~~

$u(x,b)=0$ $\quad u(x,y,0)=f(x,y)$

$u(0,y)=0$

$u(a,y)=0$

$u = XYT$

$XYT' = k(X''YT + XY''T)$

$$\frac{T'}{kT} = \frac{X''}{X} + \frac{Y''}{Y}$$

goal: want to see $X'' + \lambda X = 0$

or the Y version

$$\frac{X''}{X} = \frac{T'}{kT} - \frac{Y''}{Y} = -\lambda \quad \rightarrow \quad X'' + \lambda X = 0 \quad X(0) = X(a) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{a^2} \quad X_n = \sin\left(\frac{n\pi}{a}x\right)$$

$$\frac{Y''}{Y} = \frac{T'}{kT} + \lambda = -\mu \quad \rightarrow \quad Y'' + \mu Y = 0 \qquad Y(0) = Y(b) = 0$$

$$\mu_m = \frac{m^2\pi^2}{b^2} \qquad Y_m = \sin\left(\frac{m\pi}{b} y\right)$$

$$\frac{T'}{kT} = -(\lambda + \mu)$$

$$T' + k(\lambda + \mu)T = 0$$

$$T_{nm} = e^{-k\left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)t}$$

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm}\, e^{-k\left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)t} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right)$$

$$u(x,y,0) = f(x,y)$$

$$f(x,y) = \sum_{n=1}^{\infty} \underbrace{\left[\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{m\pi}{b} y\right)\right]}_{\text{constant if } y \text{ is fixed}} \sin\left(\frac{n\pi}{a} x\right)$$

sine series w/ $L = a$

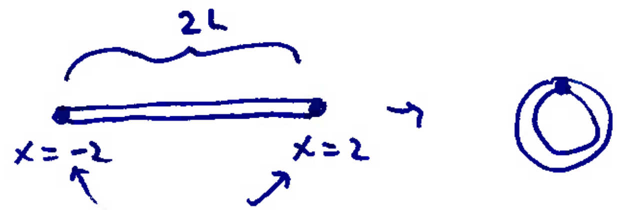
w/ constats

$$\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi}{b} y\right)$$

$$D = \sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi}{b} y\right) = \frac{2}{a}\int_0^a f(x,y)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$C_{nm} = \frac{2}{b}\int_0^b D \sin\left(\frac{n\pi}{b} y\right) dy = \frac{4}{ab}\int_0^a \int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) dy\, dx$$

\#6 practice problems



BCs must match

$2L = 4$

BCs: $u(-2, t) = u(2, t)$

$u_x(-2, t) = u_x(2, t)$

$u_t = k u_{xx} \qquad -2 < x < 2$

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \qquad X'' + \lambda X = 0$$

$$T' + k\lambda T = 0$$

$u(-2, t) = u(2, t) \rightarrow X(-2) = X(2)$

$u_x(-2, t) = u_x(2, t) \rightarrow X'(-2) = X'(2)$

$$X'' + \lambda X = 0 \qquad X = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

$$\left.\begin{aligned} X(-2) &= A\cos(-2\sqrt{\lambda}) + B\sin(-2\sqrt{\lambda}) \\ X(2) &= A\cos(2\sqrt{\lambda}) + B(\sin 2\sqrt{\lambda}) \end{aligned}\right\}\ \text{same}$$

$$-B\sin(2\sqrt{\lambda}) = B\sin(2\sqrt{\lambda})$$

$$2B\sin(2\sqrt{\lambda}) = 0$$

$$X' = -\sqrt{\lambda}A\sin(\sqrt{\lambda}\,x) + \sqrt{\lambda}B\cos(\sqrt{\lambda}\,x)$$

$$\left.\begin{aligned} X'(-2) &= -\sqrt{\lambda}A\sin(-2\sqrt{\lambda}) + \sqrt{\lambda}B\cos(-2\sqrt{\lambda}) \\ X'(2) &= -\sqrt{\lambda}A\sin(2\sqrt{\lambda}) + \sqrt{\lambda}B\cos(2\sqrt{\lambda}) \end{aligned}\right\}\ \text{same}$$

$$\sqrt{\lambda}A\sin(2\sqrt{\lambda}) = -\sqrt{\lambda}A\sin(2\sqrt{\lambda})$$

$$2A\sqrt{\lambda}\sin(2\sqrt{\lambda}) = 0$$

$\sin(2\sqrt{\lambda}) = 0$ find λ the usual way

$$2\sqrt{\lambda} = n\pi \qquad \lambda = \frac{n^2\pi^2}{4} \qquad n = 1, 2, 3, \ldots$$

$$u_t = u_{xx} \qquad 0 < x < 1$$

$$u_x(0,t) = 0 \qquad u(x,0) = f(x)$$

$$u_x(1,t) = 0$$

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \qquad X'' + \lambda X = 0 \qquad X'(0) = X'(1) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{1^2} = n^2\pi^2$$

$$X_n = \cos(n\pi x) \qquad n = 0, 1, 2, 3, \ldots$$

$$T' + n^2\pi^2 T = 0$$

$$T_n = e^{-n^2\pi^2 t}$$

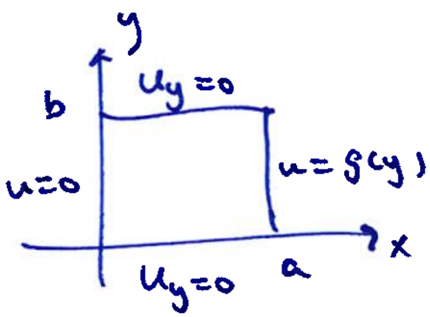
$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-n^2\pi^2 t} \cos(n\pi x)$$

at $t = 0$

$$f(x) = \sum_{n=0}^{\infty} c_n \cos(n\pi x) = \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n \cos(n\pi x)$$

artificially introduced so we have cosine series

12.



$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$Y'' + \lambda Y = 0 \qquad Y'(0) = Y'(b) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{b^2} \qquad Y_n = \cos\left(\frac{n\pi}{b} y\right)$$

$$n = 0, 1, 2, \ldots$$

$$X'' - \lambda X = 0 \qquad X(0) = 0 \qquad (\lambda \neq 0)$$

$$X = A\cosh\left(\frac{n\pi}{b} x\right) + B\sinh\left(\frac{n\pi}{b} x\right)$$

$$0 = A \qquad X_n = \sinh\left(\frac{n\pi}{b} x\right)$$

$$X'' = 0 \qquad X(0) = 0 \qquad \lambda = 0$$

$$X = Ax + B$$

$$X(0) = 0 \rightarrow B = 0 \qquad X_0 = x$$

$\cancel{u(x,y) = \sum_{n=0}^{\infty}}$